The algebraic structure of a cosmological term in spherically symmetric solutions

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We propose to describe the dynamics of a cosmological term in the spherically symmetric case by an r-dependent second rank symmetric tensor $\Lambda_{\mu\nu}$ invariant under boosts in the radial direction. This proposal is based on the Petrov classification scheme and Einstein field equations in the spherically symmetric case. The inflationary equation of state $p = -\rho$ is satisfied by the radial pressure, $p_r^\Lambda = -\rho^\Lambda$. The tangential pressure $p_\perp^\Lambda$ is calculated from the conservation equation $\Lambda^\mu_{\nu;\mu} = 0$.

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Developments in particles and quantum field theory, as well as the confrontation of models with observations in cosmology [1], compellingly favour treating the cosmological constant Λ as a dynamical quantity.

The Einstein equations with a cosmological term read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor of a matter, and Λ is the cosmological constant. In the absence of matter described by $T_{\mu\nu}$, Λ must be constant, since the Bianchi identities guarantee vanishing covariant divergence of the Einstein tensor, $G^{\mu\nu}{}_{;\nu} = 0$.

In quantum field theory, the vacuum stress-energy tensor has the form $< T_{\mu\nu} >=< \rho_{vac} > g_{\mu\nu}$ which behaves like a cosmological term with $\Lambda = 8\pi G \rho_{vac}$.

The idea that Λ might be variable has been studied for more than two decades (see [2,3] and references therein). In a recent paper on Λ-variability, Overduin and Cooperstock distinguish three approaches [4]. In the first approach $\Lambda g_{\mu\nu}$ is shifted onto the right-hand side of the field equations (1) and treated as part of the matter content. This approach, characterized by Overduin and Cooperstock as being connected to dialectic materialism, goes back to Gliner who interpreted $\Lambda g_{\mu\nu}$ as corresponding to vacuum stress-energy tensor with the equation of state $p = -\rho$ [5], to Zel'dovich who connected Λ with the gravitational interaction of virtual particles [6], and to Linde who suggested that Λ can vary [7]. In the ref. [8] a cosmological model was proposed with the equation of state varying from $p = -\rho$ to $p = \rho/3$. In contrast, idealistic approach prefers to keep Λ on the left-hand side of the Eq.(1) and treat it as a constant of nature. The third approach, allowing Λ to vary while keeping it on the left-hand side as a geometrical entity, was first applied by Dolgov in a model in which a classically unstable scalar field, non-minimally coupled to gravity, develops a negative energy density cancelling the initial positive value of a cosmological constant Λ [9].

Whenever variability of Λ is possible, it requires the presence of some matter source other than $T_{\mu\nu} = (8\pi G)^{-1}\Lambda g_{\mu\nu}$, since the conservation equation $G^{\mu\nu}{}_{;\nu} = 0$ implies $\Lambda = const$ in this case. This requirement makes it impossible to introduce a cosmological term as variable in itself. However, it is possible for a stress-energy tensor other than $\Lambda g_{\mu\nu}$.

The aim of this letter is to show what the algebraic structure of a cosmological term can be in the spherically symmetric case, as suggested by the Petrov classification scheme [10] and by the Einstein field equations.

In the spherically symmetric static case a line element can be written in the form [11]

$$ds^2 = e^{\mu(r)}dt^2 - e^{\nu(r)}dr^2 - r^2 d\Omega^2 \tag{2}$$

where $d\Omega^2$ is the line element on the unit sphere. The Einstein equations are

$$8\pi G T^t_t = e^{-\nu}(\nu'/r - 1/r^2) + 1/r^2 \tag{3}$$

$$8\pi G T^r_r = -e^{-\nu}(\mu'/r + 1/r^2) + 1/r^2 \tag{4}$$

$$8\pi G T^\theta_\theta = 8\pi G T^\phi_\phi =$$

$$-e^{-\nu}(\mu''/2 + \mu'^2/4 + (\mu' - \nu')/2r - \mu'\nu'/4) \tag{5}$$

A prime denotes differentiation with respect to r.
In the case of

$$T_{\mu\nu} = (8\pi G)^{-1}\Lambda g_{\mu\nu} = \rho_{vac} g_{\mu\nu} \tag{6}$$

the solution is the de Sitter geometry with constant positive curvature $R = 4\Lambda$. The line element is

$$ds^2 = \left(1 - \frac{\Lambda r^2}{3}\right)dt^2 - \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{7}$$

The algebraic structure of the stress-energy tensor (6), corresponding to a cosmological term $\Lambda g_{\mu\nu}$, is

$$T^t_t = T^r_r = T^\theta_\theta = T^\phi_\phi, \tag{8}$$

and the equation of state is

$$p = -\rho \tag{9}$$

In the Petrov classification scheme [10] stress-energy tensors are classified on the basis of their algebraic

structure. When the elementary divisors of the matrix $T_{\mu\nu} - \lambda g_{\mu\nu}$ (i.e., the eigenvalues of $T_{\mu\nu}$) are real, the eigenvectors of $T_{\mu\nu}$ are nonisotropic and form a comoving reference frame. Its timelike vector represents a velocity. The classification of the possible algebraic structures of stress-energy tensors satisfying the above conditions contains five possible types: [IIII],[I(III)], [II(II)], [(II)(II)], [(IIII)]. The first symbol denotes the eigenvalue related to the timelike eigenvector. Parentheses combine equal (degenerate) eigenvalues. A comoving reference frame is defined uniquely if and only if none of the spacelike eigenvalues $\lambda_\alpha(\alpha = 1, 2, 3)$ coincides with a timelike eigenvalue λ_0. Otherwise there exists an infinite set of comoving reference frames.

In this scheme the de Sitter stress-energy tensor (6) is represented by [(IIII)] (all eigenvalues being equal) and classified as a vacuum tensor due to the absence of a preferred comoving reference frame. An observer moving through the de Sitter vacuum (6) cannot in principle measure his velocity with respect to it, since his comoving reference frame is also comoving for (6) [5].

In the spherically symmetric case it is possible, by the same definition, to introduce an r-dependent vacuum stress-energy tensor with the algebraic structure [12]

$$T^t_t = T^r_r; \quad T^\theta_\theta = T^\phi_\phi \tag{10}$$

In the Petrov classification scheme this stress-energy tensor is denoted by [(II)(II)]. It has an infinite set of comoving reference frames, since it is invariant under rotations in the (r,t) plane. Therefore an observer moving through it cannot in principle measure the radial component of his velocity. The stress-energy tensor (10) describes a spherically symmetric anisotropic vacuum invariant under the boosts in the radial direction [12].

The conservation equation $T^{\mu\nu}_{\ \ ;\nu} = 0$ gives the r-dependent equation of state [13,12]

$$p_r = -\rho; \quad p_\perp = p_r + (r/2)(dp_r/dr) \tag{11}$$

where $\rho = T^t_t$ is the density, $p_r = -T^r_r$ is the radial pressure, and $p_\perp = -T^\theta_\theta = -T^\phi_\phi$ is the tangential pressure. In this case equations (3)-(4) reduce to the equation

$$8\pi G\rho = e^{-\nu(r)}(\nu'/r - 1/r^2) + 1/r^2 \tag{12}$$

whose solution is

$$g_{00} = e^{-\nu(r)} = 1 - \frac{2G\mathcal{M}(r)}{r}; \quad \mathcal{M}(r) = \int_0^r \rho(x)x^2 dx \tag{13}$$

and the line element is

$$ds^2 = (1-2G\mathcal{M}(r)/r)dt^2 - (1-2G\mathcal{M}(r)/r)^{-1}dr^2 - r^2 d\Omega^2 \tag{14}$$

If we require the density $\rho(r)$ to vanish as $r \to \infty$ quicker then r^{-3}, then the metric (14) for large r has the Schwarzschild form

$$g_{00}(r) = 1 - 2GM/r \tag{15}$$

with

$$M = 4\pi \int_0^\infty \rho(r)r^2 dr < \infty \tag{16}$$

If we impose the boundary condition of de Sitter behaviour (7) at $r \to 0$, the form of the mass function $\mathcal{M}(r)$ in the limit of small r must be [13–15]

$$\mathcal{M}(r) = (\Lambda/6G)r^3 = (4\pi/3)\rho_{vac}r^3 \tag{17}$$

For any density profile satisfying conditions (16)-(17), the metric (14) describes a globally regular de Sitter-Schwarzschild geometry, asymptotically Schwarzschild as $r \to \infty$ and asymptotically de Sitter as $r \to 0$ [15,16].

The fundamental difference from the Schwarzschild case is that there are two horizons, a black hole horizon r_+ and an internal Cauchy horizon r_- [13,14,12]. A critical value of the mass M_{crit} exists, at which the horizons come together. This gives a lower limit for the black hole mass.

Depending on the value of the mass M, there exist three types of configurations in which a Schwarzschild singularity is replaced with Λ core [15,16]: 1) A Λ black hole (ΛBH) for $M > M_{crit}$ [17]; 2) An extreme ΛBH for $M = M_{crit}$; 3) A "Λ particle" (ΛP) - a particle-like structure without horizons "made up" of a self-gravitating spherically symmetric vacuum (10) - for $M < M_{crit}$.

In the course of Hawking evaporation, a ΛBH loses its mass and the configuration evolves towards a ΛP [15].

De Sitter-Schwarzschild configurations are plotted in Fig.1 for the case of the density profile [12,15]

$$\rho(r) = 8\pi G\Lambda \exp\left(-\frac{\Lambda}{6GM}r^3\right) = \rho_{vac}\exp\left(-\frac{4\pi}{3}\frac{\rho_{vac}}{M}r^3\right) \tag{18}$$

The mass function in the metric (14) then takes the form

$$\mathcal{M}(r) = M\left(1 - \exp\left(-\frac{\Lambda}{6GM}r^3\right)\right) \tag{19}$$

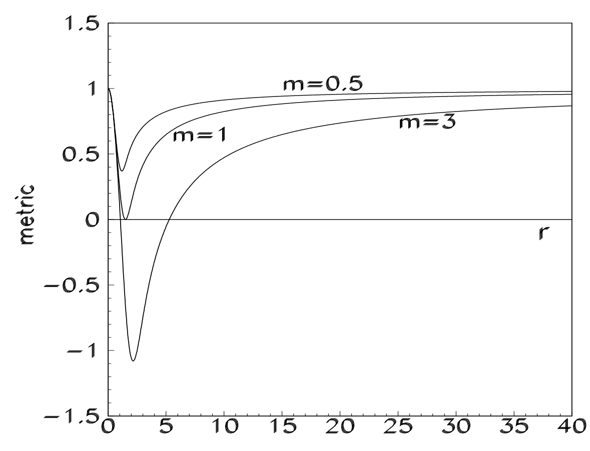

FIG. 1. The metric coefficient $g_{00}(r)$ for de Sitter-Schwarzschild configurations in the case of the density profile (18). The parameter m is the mass M normalized to $M_{crit} \simeq 0.3 M_{Pl}(\rho_{Pl}/\rho_\Lambda)^{1/2}$.

The stress-energy tensor (10) responsible for the ΛBH and ΛP solutions connects in a smooth way two vacuum states: de Sitter vacuum (6) at the origin and Minkowski vacuum $T_{\mu\nu} = 0$ at infinity. The vacuum equation of state (9) remains valid for the radial component of a pressure. This makes it possible to treat the stress-energy tensor (10) as corresponding to an r-dependent cosmological term $\Lambda_{\mu\nu}$, varying from $\Lambda_{\mu\nu} = \Lambda g_{\mu\nu}$ as $r \to 0$ to $\Lambda_{\mu\nu} = 0$ as $r \to \infty$, and satisfying the equation of state (11) with $\rho^{\Lambda} = \Lambda_t^t$, $p_r^{\Lambda} = -\Lambda_r^r$ and $p_{\perp}^{\Lambda} = -\Lambda_{\theta}^{\theta} = -\Lambda_{\phi}^{\phi}$.

The global structure of de Sitter-Schwarzschild spacetime in the case $M > M_{crit}$ is shown in Fig.2 [15]. It contains an infinite sequence of Λ black holes ($\mathcal{BH}$), Λ white holes ($\mathcal{WH}$), past and future Λ cores ($\mathcal{RC}$), and asymptotically flat universes ($\mathcal{U}$). A Λ white hole models a non-singular cosmology with inflationary origin followed by anisotropic Kasner-type expansion due to the anisotropy of the Λ tensor, which in this model is time-dependent [23].

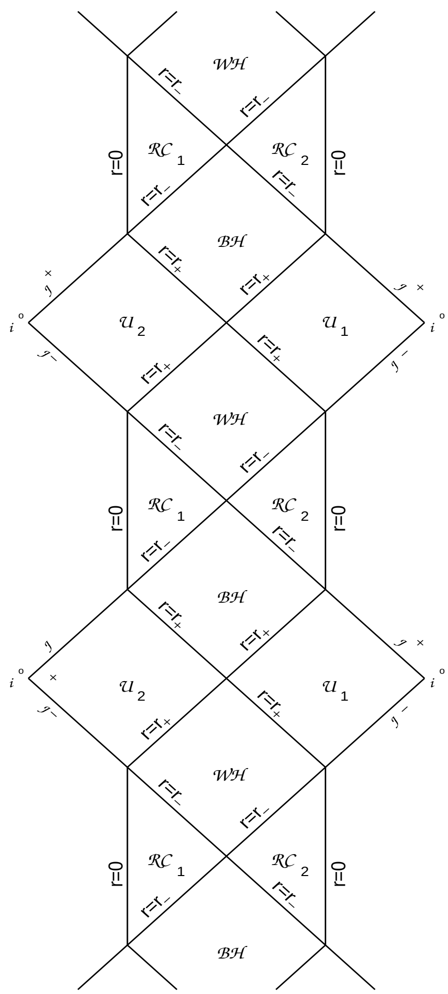

FIG. 2. Penrose-Carter diagram for Λ black hole.

The conformal diagram shown in Fig.2 represents the global structure of de Sitter-Schwarzschild spacetime in the case of any smooth density profile $\rho(r)$ satisfying conditions (16)-(17). In the case of discontinuous density profile $\rho(r) = \rho_{Pl}\Theta(r_m - r)$ corresponding to direct matching of de Sitter to Schwarzschild metric at the junction surface $r = r_m$ [13,14], the asymptotically flat regions are connected through a black hole interior to a baby universe arising inside a black hole [14]. The question of arising a baby universe inside a black hole in the case of an arbitrary continuous density profile is considered in the ref. [23].

The vacuum energy outside a ΛBH horizon is given by

$$E_{vac} = \int_{r_+}^{\infty} \rho(r) r^2 dr = M \exp\left(-\frac{\Lambda}{6GM} r_+^3\right) \qquad (20)$$

One can say that a Λ black hole has Λ hair.

The question of the stability of a ΛBH and ΛP is currently under investigation. Comparison of the ADM mass (16) with the proper mass [18]

$$\mu = 4\pi \int_0^{\infty} \rho(r)(1 - 2G\mathcal{M}(r)/r)^{-1/2} r^2 dr \qquad (21)$$

makes a suggestion. In the spherically symmetric situations the ADM mass represents the total energy, $M = \mu + \textit{binding energy}$ [18]. In our case μ is bigger than M. This gives us a hint that the configuration might be stable since energy is needed to break it up.

If we modify the density profile to allow a non-zero value of cosmological constant λ as $r \to \infty$, putting

$$T_t^t(r) = \rho(r) + 8\pi G\lambda, \qquad (22)$$

we obtain the metric [19]

$$ds^2 = (1 - 2G\mathcal{M}(r)/r - \lambda r^2/3) dt^2 -$$

$$(1 - 2G\mathcal{M}(r)/r - \lambda r^2/3)^{-1} dr^2 - r^2 d\Omega^2 \qquad (23)$$

whose asymptotics are the de Sitter metric (7) with $(\Lambda + \lambda)$ as $r \to 0$ and with λ as $r \to \infty$. The two-lambda spacetime has in general three horizons: a cosmological horizon r_{++}, a black hole horizon r_+ and a Cauchy horizon r_-. Horizons are calculated by solving the equation $g_{00}(r) = 0$ with $g_{00}(r)$ from the Eq.(23). They are plotted in Fig.3 for the case of the density profile given by (18). There are two critical values of the mass M, restricting the BH mass from below and above. A lower limit M_{cr1} corresponds to the first extreme BH state $r_+ = r_-$ and is very close to the lower limit for ΛBH. An upper limit M_{cr2} corresponds to the second extreme state $r_+ = r_{++}$ and depends on the parameter $q \equiv \sqrt{\Lambda/\lambda}$.

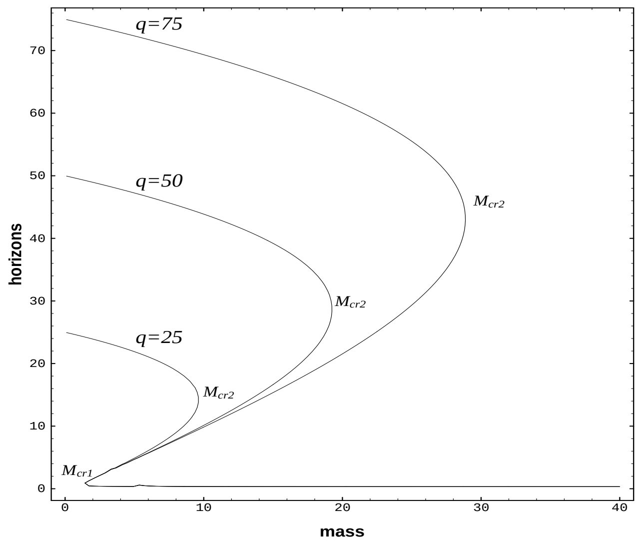

FIG. 3. Horizon-mass diagram forn a two-lambda spacetime. An upper limit for a BH mass M_{cr2} depends on the parameter $q \equiv (\Lambda/\lambda)^{1/2}$.

Depending on the mass M, two-lambda geometries represent five types of configurations [20]:
1) A two-lambda black hole ($\Lambda\lambda$BH) for $M_{cr1} < M < M_{cr2}$, which is a non-singular cosmological black hole, i.e. a non-singular modification of the Kottler-Trefftz solution [21], frequently referred to in the literature as a Schwarzschild-de Sitter black hole. 2) An extreme $\Lambda\lambda$BH with the minimum possible mass $M = M_{cr1}$. 3)A ΛP with a de Sitter background of small λ for $M < M_{cr1}$. 4) An extreme $\Lambda\lambda$BH with the maximum possible mass $M = M_{cr2}$, which is the non-singular modification of the Nariai solution [22]. 5) Soliton-like configuration for $M > M_{cr2}$, a one-horizon solution with lambda varying from Λ at the origin to λ at infinity, which can be called a "Λ-bag". These configurations are plotted in Fig.4.

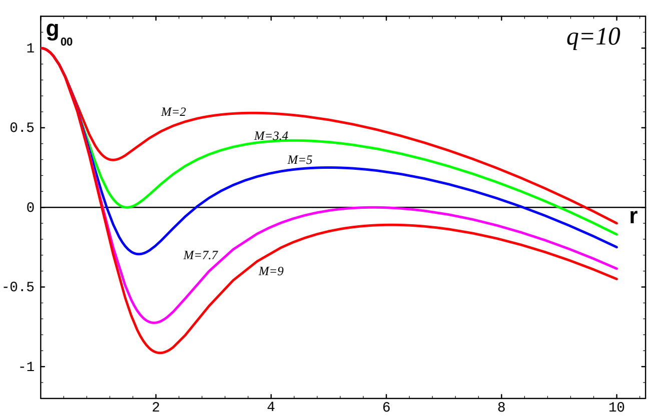

FIG. 4. Two-lambda configurations for the case $q = 10$. The mass M is normalized to $(3/G^2\Lambda)^{1/2}$. Two extreme states for $\Lambda\lambda$BH are $M_{cr1} \simeq 3.4$ and $M_{cr2} \simeq 7.7$.

The global structure of two-lambda spacetimes for the case $M_{cr1} < M < M_{cr2}$ is shown in Fig.5. It contains an infinite sequence of $\Lambda\lambda$BH, $\Lambda\lambda$WH (white holes with Λ at the origin and λ at infinity), Λ future and past cores, and asymptotically de Sitter universes with small λ. A two-lambda white hole models non-singular cosmology with inflationary origin followed by anisotropic Kasner-like stage and ended in λ dominated stage [24].

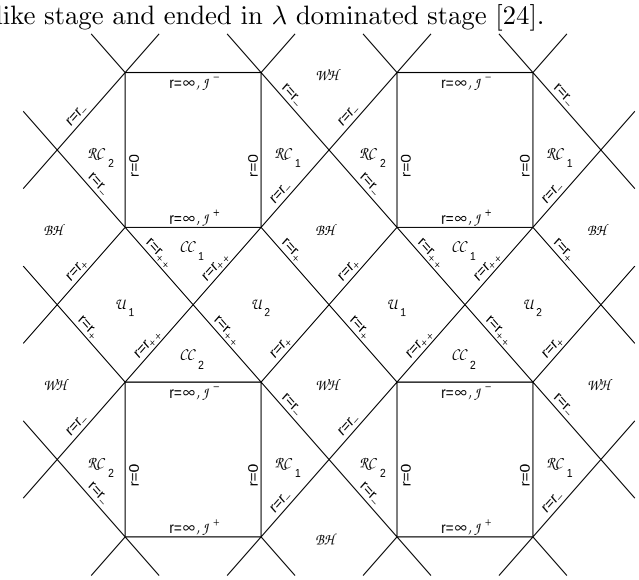

FIG. 5. Penrose-Carter diagram for $\Lambda\lambda$ black holes. There is an infinite sequence of black and white holes $\mathcal{BH}$,$\mathcal{WH}$, whose singularities are replaced by Λ cores $\mathcal{RC}_1$,$\mathcal{RC}_2$, asymptotically de Sitter universes $\mathcal{U}_1$,$\mathcal{U}_2$, and λ cores $\mathcal{CC}_1$,$\mathcal{CC}_2$ (regions beyond the cosmological horizons r_{++}).

The stress-energy tensor responsible for the two-lambda geometry connects in a smooth way two vacuum states with non-zero cosmological constant: de Sitter vacuum $T_{\mu\nu} = (8\pi G)^{-1}(\Lambda+\lambda)g_{\mu\nu}$ at the origin, and de Sitter vacuum $T_{\mu\nu} = (8\pi G)^{-1}\lambda g_{\mu\nu}$ at infinity. This confirms the proposed interpretation of the stress-energy tensor (10) as corresponding to a variable effective cosmological term $\Lambda_{\mu\nu}$.

In conclusion, let us compare the proposed variable cosmological tensor $\Lambda_{\mu\nu}$ with the quintessence which is a time-varying, spatially inhomogeneous component of matter content with negative pressure [25]. The key difference comes from the equation of state. For quintessence the equation of state is $p = -\alpha\rho$ with $\alpha < 1$ [25]. This corresponds to such a stress-energy tensor $T_{\mu\nu}$ for which a comoving reference frame is defined uniquely. The quintessence represents thus a non-vacuum alternative to a cosmological constant Λ, while the tensor $\Lambda_{\mu\nu}$ represents the extension of the algebraic structure of a cosmological term which allows it to be variable.

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